Limit & Continuity :-

$$\boxed{RHL = LHL}$$

$f(a)^+ = f(a)^-$ limit exist

$f(x) = x^2+1$

$f(2) = 2^2+1 = 5$

R.H.L $\lim_{h \to 0} f(2^+) = (2+h)^2+1 = 5$

L.H.L $\lim_{h \to 0} f(2^-) = (2-h)^2+1 = 5$

So, it is Continuity

Partial Differentiation

$$f = x^2 + y^2 + 2xy + 1$$

$$U_x = f(x) = \frac{\partial f}{\partial x} = 2x + 0 + 2y$$

$$U_y = f(y) = \frac{\partial f}{\partial y} = 0 + 2y + 2x + 0$$

$$f(xx) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial}{\partial x}\left[2x + 2y\right] = 2$$

$$f(yy) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial}{\partial y}\left[2y + 2x\right] = 2$$

$$f(xy) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial}{\partial x}\left[2y + 2x\right] = 2$$

$$f(yx) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial}{\partial y}\left[2x + 2y\right] = 2$$

Ques :- $U = \log\sqrt{x^2+y^2+z^2}$ — (1)

Prove that $(x^2+y^2+z^2)[U_{xx}+U_{yy}+U_{zz}]$

Eq^n (1) w.r.t. x P.diff

$$U_x = \frac{\partial u}{\partial x} = \frac{1}{\sqrt{x^2+y^2+z^2}} \times \frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2}$$

$$U_x = \frac{1}{\sqrt{x^2+y^2+z^2}} \times \frac{1}{2\sqrt{x^2+y^2+z^2}} \times \frac{\partial}{\partial x}(x^2+y^2+z^2)$$

$$U_x = \frac{1}{\cancel{2}(x^2+y^2+z^2)} \times \cancel{2}x$$

$$U_x = \frac{x}{x^2+y^2+z^2}$$

$$U_{xx} = \frac{\partial U_x}{\partial x} = \frac{(x^2+y^2+z^2) - x\,2x}{(x^2+y^2+z^2)^2} = \frac{y^2+z^2-x^2}{(x^2+y^2+z^2)^2} \quad (2)$$

Similarly :-

$$U_{yy} = \frac{x^2+z^2-y^2}{(x^2+y^2+z^2)^2} \quad (3) \qquad U_{zz} = \frac{x^2+y^2-z^2}{(x^2+y^2+z^2)^2} \quad (4)$$

Apply :-

$U_{xx} + U_{yy} + U_{zz} =$

$$\Rightarrow \frac{y^2 + z^2 - \cancel{x^2} + x^2 + \cancel{z^2} - \cancel{y^2} + x^2 + \cancel{y^2} - \cancel{z^2}}{(x^2 + y^2 + z^2)^2}$$

$$= \frac{\cancel{x^2 + y^2 + z^2}}{(x^2 + y^2 + z^2)^{\cancel{2}}}$$

$$U_{xx} + U_{yy} + U_{zz} = \frac{1}{x^2 + y^2 + z^2}$$

$$(x^2 + y^2 + z^2)\, U_{xx} + U_{yy} + U_{zz} = 1$$

Ques:- $V = (x^2 - y^2) f(xy)$

Proof:- $V_{xx} + V_{yy} = (x^4 - y^4) f''(xy)$

$V_{(x)}$ $\frac{\partial V}{\partial x} = 2x\, f(xy) + (x^2 - y^2) f'(xy) \cdot y$

$= 2x f(xy) + (x^2 - y^2) f'(xy) \cdot y$

again:- w.r.t x

$$\frac{\partial^2 V}{\partial x^2} = 2 \cdot f(xy) + 2xy\, f'(xy) + 2yx\, f'(xy) + (x^2 - y^2) f''(xy)\, y^2$$

~~Similarly~~:-

$V_y = \frac{\partial V}{\partial y} = (x^2 - y^2) f'(xy) \cdot x + f(xy)(-2y)$

$= x(x^2 - y^2) f'(xy) - 2y\, f(xy)$

$V_{yy} = \frac{\partial V_y}{\partial y} = -2f(xy) - 4xy f'(xy) + (x^2 - y^2) x^2 f''(xy)$

Now:-

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = (x^2 - y^2) f''(xy) [y^2 + x^2]$$

$$= [x^4 - y^4] f''(xy)$$

Ques:- $u = e^{xyz}$ find the value $\frac{\partial^3 v}{\partial x \partial y \partial z}$

$$\frac{\partial^3 v}{\partial x \partial y \partial z} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left[\frac{\partial v}{\partial z}\right]\right]$$

$$\Rightarrow \frac{\partial v}{\partial z} = e^{xyz} \cdot xy$$

$$\Rightarrow \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial z}\right] = x \cdot \frac{\partial}{\partial y}\left[y \cdot e^{xyz}\right]$$

$$= x \cdot \left[1 \cdot e^{xyz} + xyz e^{xyz}\right]$$

$$\Rightarrow \frac{\partial^2 u}{\partial y \partial z} = (x + x^2yz) e^{xyz}$$

$$\Rightarrow \frac{\partial^3 u}{\partial x \partial y \partial z} = \frac{\partial u}{\partial x}\left[\frac{\partial^2 u}{\partial y \partial z}\right] = \frac{\partial}{\partial x}\left[(x + x^2yz) e^{xyz}\right]$$

$$= (1 + 2xyz) e^{xyz} + \left\{(x + x^2yz) e^{xyz} yz\right.$$

$$= \left[1 + 2xyz + xyz + x^2y^2z^2\right] e^{xyz}$$

$$= \left[1 + 3xyz + x^2y^2z^2\right] e^{xyz}$$

Maths-3

Eular's theorem :-

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu \quad \text{(i)}$$

n= degree of the function

$$\boxed{f=\frac{\sqrt{x}+\sqrt{y}}{x^2+y^2}}$$

Ques :- $u=\sin^{-1}\left[\frac{x^3+y^3+z^3}{ax+by+cz}\right]$

Prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=2\tan u$

Soln:-

$$u=\sin^{-1}\left[\frac{x^3+y^3+z^3}{ax+by+cz}\right]$$

let, $v = \sin u=\frac{x^3+y^3+z^3}{ax+by+cz}$

v is a homogeneous function order 2

By Eular's function →

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}+z\frac{\partial v}{\partial z}=2v$$

Put the value of v

$$x\frac{\partial \sin u}{\partial x}+y\frac{\partial \sin u}{\partial y}+z\frac{\partial \sin u}{\partial z}=2\sin u$$

$$\Rightarrow x \cos u \frac{\partial u}{\partial x} + y \cos u \frac{\partial u}{\partial y} + z \cos u \frac{\partial u}{\partial z} = 2 \sin$$

$$\cancel{\Rightarrow x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z}}$$

$$\Rightarrow x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 2 \frac{\sin u}{\cos u} = 2 \tan$$

Ques. :⇒

$$U = \tan^{-1}\frac{x^3+y^3}{x-y}$$

Prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$$

or

$$x^2\frac{\partial^2 u}{\partial x} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{2\cos 3u}{\sin u}$$

Soln :-

$$U = \tan^{-1}\frac{x^3+y^3}{x-y}$$

$$V = \tan u = \frac{x^3+y^3}{x-y}$$

V is the homogeneous function order (2)

By Euler's theorem :→

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 2v$$

$$x\frac{\partial \tan u}{\partial x} + y\frac{\partial \tan u}{\partial y} = 2\tan u$$

$$x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = 2\tan u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\tan u}{\sec^2 u}$$

Ques:-

$$U = \overset{z}{\sin^{-1}\frac{x}{y}} + \overset{\alpha}{\tan^{-1}\frac{y}{x}}$$

$$z = \sin^{-1}\frac{x}{y} \Rightarrow \sin z = \frac{x}{y} \quad \text{—(A)}$$

$$\alpha = \tan^{-1}\frac{y}{x} \Rightarrow \tan\alpha = \frac{y}{x} \quad \text{—(B)}$$

$$\Rightarrow x\frac{\partial A}{\partial x} + y\frac{\partial A}{\partial y} = 0.A$$

$$\Rightarrow x\frac{\partial \sin z}{\partial x} + y\frac{\partial \sin z}{\partial y} = 0$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0 \quad \text{—(1)}$$

$$\Rightarrow x\frac{\partial B}{\partial x} + y\frac{\partial B}{\partial y} = 0.\alpha$$

$$x\frac{\partial \tan\alpha}{\partial x} + y\frac{\partial \tan\alpha}{\partial y} = 0$$

$$x\frac{\partial \alpha}{\partial x} + y\frac{\partial \alpha}{\partial y} = 0 \quad \text{—(2)}$$

Now:- $z + \alpha = U$

$$x\frac{\partial}{\partial x}(z+\alpha) + y\frac{\partial}{\partial y}(z+\alpha) = 0$$

$$\boxed{x\frac{\partial U}{\partial x} + y\frac{\partial U}{\partial y} = 0}$$

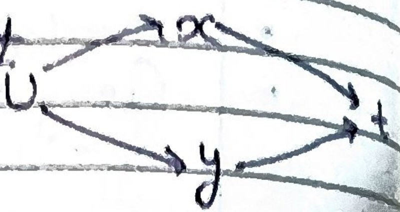
Maths 3

1. **Total Derivative / Partial Breaking**

Ques. 1

$U = \sin^{-1}(x-y)$ $\qquad x = 3t, \; y = 4t^3$

$$\frac{dU}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$



$$= \frac{1}{\sqrt{1-(x-y)^2}} \times 3 + \frac{1}{\sqrt{1-(x-y)^2}}(-1) \times 12t^2$$

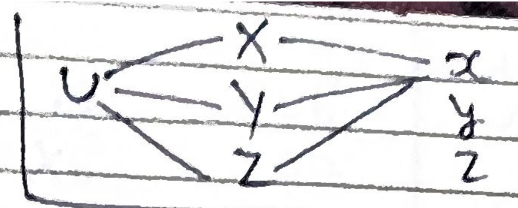
$$= \frac{1}{\sqrt{1-(x-y)^2}}\left[3 - 12t^2\right]$$

$$= \frac{3(1-4t^2)}{\sqrt{1-(3t-4t^3)^2}}$$

One variable $= \frac{d}{dx}$

more than one variable $= \frac{\partial}{\partial x}$

$\frac{\partial}{\partial x}\sin^{-1}x = \frac{1}{\sqrt{x}}$



Ques. 2

$U = f[(y-z), (z-x), (x-y)]$

Prove that, $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Sol^n:-

let, $X = y-z, \; Y = z-x, \; Z = x-y$

$U = f(X, Y, Z)$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial u}{\partial Y}\frac{\partial Y}{\partial x} + \frac{\partial u}{\partial Z}\frac{\partial Z}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial F}{\partial X} \times 0 + \frac{\partial F}{\partial Y}(-1) + \frac{\partial F}{\partial Z}(1)$$

$$\frac{\partial u}{\partial x} = -\frac{\partial f}{\partial Y} + \frac{\partial f}{\partial Z} \quad \text{--- (1)}$$

$$\Rightarrow \frac{\partial u}{\partial y} = \frac{\partial u}{\partial X}\frac{\partial X}{\partial y} + \frac{\partial u}{\partial Y}\frac{\partial Y}{\partial y} + \frac{\partial u}{\partial Z}\frac{\partial Z}{\partial y}$$

$$= \frac{\partial F}{\partial X}(1) + \frac{\partial F}{\partial Y}(0) + \frac{\partial F}{\partial Z}(-1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial F}{\partial X} - \frac{\partial F}{\partial Z} \quad \text{--- (2)}$$

Similarly :-

$$\frac{\partial u}{\partial z} = -\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \quad \text{--- (3)}$$

Now :->

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$$

$$\Rightarrow \cancel{-\frac{\partial f}{\partial y}} + \cancel{\frac{\partial f}{\partial z}} + \cancel{\frac{\partial f}{\partial x}} - \cancel{\frac{\partial f}{\partial z}} - \cancel{\frac{\partial f}{\partial x}} + \cancel{\frac{\partial f}{\partial y}}$$

$$\Rightarrow 0$$

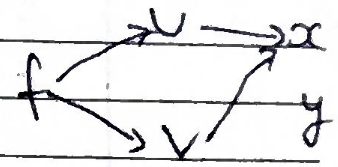
Ques. 3 $u = x^2 - y^2$, $v = 2xy$

$$f(x,y) = \theta(u,v)$$

Prove that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4(x^2+y^2)\left[\frac{\partial^2 \theta}{\partial u^2} + \frac{\partial^2 \theta}{\partial v^2}\right]$

Sol^n :->

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}$$



$$= \frac{\partial \theta}{\partial u}(2x) + \frac{\partial \theta}{\partial v}(2y)$$

$$= 2x\frac{\partial \theta}{\partial u} + 2y\frac{\partial \theta}{\partial v}$$

$$= 2\left[x\frac{\partial \theta}{\partial u} + y\frac{\partial \theta}{\partial v}\right]$$

$$\frac{\partial f}{\partial x} = 2\left[x\frac{\partial}{\partial u} + y\frac{\partial}{\partial v}\right]\theta$$

again diff. $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] = \cancel{\frac{\partial}{\partial x}}\left[\cancel{2\left[x\frac{\partial}{\partial u} + y\frac{\partial}{\partial v}\right]\theta}\right]$

$$= 2\left[x\frac{\partial}{\partial u} + y\frac{\partial}{\partial v}\right] 2\left[x\frac{\partial}{\partial u} + y\frac{\partial}{\partial u}\right]\theta$$

$$\frac{\partial^2 f}{\partial x^2} \Rightarrow 4\left[x^2\frac{\partial^2}{\partial u^2} + xy\frac{\partial^2}{\partial u\,\partial v} + xy\frac{\partial^2}{\partial v\,\partial u} + y^2\frac{\partial^2}{\partial v^2}\right]\theta$$

$$\frac{\partial^2 f}{\partial x^2} = 4\left[x^2\frac{\partial^2\theta}{\partial u^2} + 2xy\frac{\partial^2\theta}{\partial u\partial v} + y^2\frac{\partial^2\theta}{\partial v^2}\right] \text{ --- } (1)$$

Now, find $\frac{\partial^2 f}{\partial y^2}$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}$$

$$= \frac{\partial\theta}{\partial u}(-2y) + \frac{\partial\theta}{\partial v}(2x) = 2\left[-y\frac{\partial\theta}{\partial u} + x\frac{\partial\theta}{\partial v}\right]$$

$$\Rightarrow \quad \frac{\partial f}{\partial y} = 2\left[-y\frac{\partial}{\partial u} + x\frac{\partial}{\partial v}\right]\theta$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right] = 2\left[-y\frac{\partial}{\partial u} + x\frac{\partial}{\partial v}\right]2\left[-y\frac{\partial}{\partial u} + x\frac{\partial}{\partial v}\right]\theta$$

$$\frac{\partial^2 f}{\partial y^2} = 4\left[y^2\frac{\partial^2\theta}{\partial u^2} - 2xy\frac{\partial^2\theta}{\partial u\,\partial v} + x^2\frac{\partial^2\theta}{\partial v^2}\right] \text{ --- } (2)$$

Now Eq^n (1) + Eq^n (2)

$$\Rightarrow \quad 4\left[x^2\frac{\partial^2\theta}{\partial u^2} + \cancel{2xy\frac{\partial^2\theta}{\partial u\partial v}} + y^2\frac{\partial^2\theta}{\partial v^2}\right] +$$

$$+ 4\left[y^2\frac{\partial^2\theta}{\partial u^2} - \cancel{2xy\frac{\partial^2\theta}{\partial u\partial v}} + x^2\frac{\partial^2\theta}{\partial v^2}\right]$$

$$\Rightarrow \quad 4\left[(x^2+y^2)\left[\frac{\partial^2\theta}{\partial u^2} + \frac{\partial^2\theta}{\partial v^2}\right]\right]$$